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Pearson Edexcel
Level 3 GCE

Centre Number | Candidate Number

Further Mathematics

Advanced
Further Mathematics Option 1
Paper 3: Further Pure Mathematics 1

Sample Assessment Material for first teaching September 2017

Time: 1 hour 30 minutes

Paper Reference **9FM0/3A**

You must have:
Mathematical Formulae and Statistical Tables, calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

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Answer ALL questions. Write your answers in the spaces provided.

1. Use Simpson's Rule with 6 intervals to estimate

$$\int_1^4 \sqrt{1 + x^3}\,dx$$

(5)

x	1	1.5	2	2.5	3	3.5	4
y	$\sqrt{2}$	$\sqrt{\frac{35}{8}}$	3	$\sqrt{\frac{133}{8}}$	$\sqrt{28}$	$\sqrt{\frac{351}{8}}$	$\sqrt{65}$

$\text{Area} \simeq \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6)$

$h = \frac{4-1}{6}$

$= \frac{1}{2}$

$\therefore \text{Area} \simeq \frac{\frac{1}{2}}{3}\left[\sqrt{2} + 4\sqrt{\frac{35}{8}} + 2(3) + 4\sqrt{\frac{133}{8}} + 2\sqrt{28} + 4\sqrt{\frac{351}{8}} + \sqrt{65}\right]$

$\simeq \boxed{12.9}$

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Question 1 continued

(Total for Question 1 is 5 marks)

2. Given k is a constant and that

$$y = x^3 e^{kx}$$

use Leibnitz theorem to show that

$$\frac{d^n y}{dx^n} = k^{n-3}e^{kx}\left(k^3x^3 + 3nk^2x^2 + 3n(n-1)kx + n(n-1)(n-2)\right)$$

(4)

$y = x^3 e^{kx}$

$u = e^{kx}$ $\qquad v = x^3$

$u' = ue^{kx}$ $\qquad v' = 3x^2$

$u'' = u^2 e^{kx}$ $\qquad v'' = 6x$

$\dots$ $\qquad v''' = 6$

$\qquad v'''' = 0$

$$\frac{d^n y}{dx^n} = \sum_{i=0}^{n} \binom{n}{i} (u)^{n-i} (v)^i$$

$$= x^3 (e^{kx})^n + 3nx^2 (e^{kx})^{n-1} + 3nx(n-1)(e^{kx})^{n-2} +$$

$$\frac{n(n-1)(n-2)}{6}(6)(e^{kx})^{n-3}$$

but suppose $a = e^{kx}$

then $\frac{d^n a}{dx^n} = (k^n) e^{kx}$

so... $\frac{d^n y}{dx^n} = x^3(k^n e^{kx}) + 3nx^2(k^{n-1}e^{kx}) + 3nx(n-1)(k^{n-2}e^{kx})$

$+ n(n-1)(n-2)(k^{n-3}e^{kx})$

$= k^{n-3}e^{kx}\left[k^3x^3 + 3nx^2(k)^2 + 3nx(n-1)k + n(n-1)(n-2)\right]$

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Question 2 continued

(Total for Question 2 is 4 marks)

3. A vibrating spring, fixed at one end, has an external force acting on it such that the centre of the spring moves in a straight line. At time t seconds, $t \geqslant 0$, the displacement of the centre C of the spring from a fixed point O is x micrometres.

The displacement of C from O is modelled by the differential equation

$$t^2 \frac{d^2x}{dt^2} - 2t\frac{dx}{dt} + (2 + t^2)x = t^4 \qquad \text{(I)}$$

(a) Show that the transformation $x = tv$ transforms equation (I) into the equation

$$\frac{d^2v}{dt^2} + v = t \qquad \text{(II)}$$

(5)

(b) Hence find the general equation for the displacement of C from O at time t seconds.

(7)

(c) (i) State what happens to the displacement of C from O as t becomes large.

(ii) Comment on the model with reference to this long term behaviour.

(2)

a) $t^2\left(\frac{d^2x}{dt^2}\right) - 2t\left(\frac{dx}{dt}\right) + (2+t^2)x = t^4$

$x = tv \rightarrow \frac{dx}{dt} = v + t\frac{dv}{dt}$ — 1

$\therefore \frac{d^2x}{dt^2} = \frac{dv}{dt} + \frac{dv}{dt} + t\frac{d^2v}{dt^2}$

$= 2\frac{dv}{dt} + t\frac{d^2v}{dt^2}$ — 2

sub 1 and 2 into I :

$t^2\left(2\frac{dv}{dt} + t\frac{d^2v}{dt^2}\right) - 2t\left(v + t\frac{dv}{dt}\right) + (2+t^2)x = t^4$

$\therefore \cancel{2t^2\left(\frac{dv}{dt}\right)} + t^3\left(\frac{d^2v}{dt^2}\right) - 2vt - \cancel{2t^2\left(\frac{dv}{dt}\right)} + (2+t^2)x = t^4$

$\Rightarrow t^3\left(\frac{d^2v}{dt^2}\right) - 2tv + 2x + t^2x = t^4$

$x = tv \quad \therefore t^3\left(\frac{d^2v}{dt^2}\right) - \cancel{2x} + \cancel{2x} + t^3v = t^4$

$\div t^3$ $\frac{d^2v}{dt^2} + v = t$

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Question 3 continued

b) AUX : $\lambda^2 + 1 = 0$

$\lambda^2 = -1$ $\therefore \lambda = \pm i$

CF : $v = A\sin t + B\cos t$

PI : let $v = \lambda t + \gamma$

then $\dot{v} = \lambda$

and $\ddot{v} = 0$

subbing back : $0 + \lambda t + \gamma = t$

comparing coefficients : $\lambda = 1$

$\gamma = 0$

so general solution : $v = A\sin t + B\cos t + t$

but we want a general equation for x (displacement)

$x = vt$: $\boxed{x = t(A\sin t + B\cos t + t)}$

ci) as t becomes large, x also becomes very large

ii) The issue is the model suggests with large t, x will be large and keep increasing. Not realistic as spring would probably break eventually.

(Total for Question 3 is 14 marks)

4. $$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + y = 0 \qquad \text{(I)}$$

(a) Show that

$$\frac{d^5y}{dx^5} = ax\frac{d^4y}{dx^4} + b\frac{d^3y}{dx^3}$$

where a and b are integers to be found.

(4)

(b) Hence find a series solution, in ascending powers of x, as far as the term in x^5, of the differential equation (I) where $y = 0$ and $\frac{dy}{dx} = 1$ at $x = 0$

(5)

a) $y'' - 2xy' + y = 0$

$\Rightarrow y''' - 2y' - 2xy'' + y' = 0$

$\therefore y''' - y' - 2xy'' = 0$

$\Rightarrow y'''' - y'' - 2y'' - 2xy''' = 0$

$\therefore y'''' - 3y'' - 2xy''' = 0$

$\Rightarrow y''''' - 3y''' - 2y''' - 2xy'''' = 0$

$\therefore y''''' = 5y''' + 2xy''''$

hence $\boxed{\frac{d^5y}{dx^5} = 5\frac{d^3y}{dx^3} + 2x\frac{d^4y}{dx^4}}$

b) $x = 0$, $y = 0$, $y' = 1$

$y'' = 2(0)(1) + 0 = 0$

$y''' = 1 + 2(0)(0) = 1$

$y'''' = 3(0) + 0 = 0$

$y''''' = 5(1) + 0 = 5$

so $y \simeq 0 + 1x + 0\left(\frac{x^2}{2}\right) + 1\left(\frac{x^3}{6}\right) + 0\left(\frac{x^4}{24}\right) + 5\left(\frac{x^5}{120}\right)$

hence $\boxed{y \simeq x + \frac{x^3}{6} + \frac{x^5}{24}}$

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Question 4 continued

(Total for Question 4 is 9 marks)

5. The normal to the parabola $y^2 = 4ax$ at the point $P(ap^2, 2ap)$ passes through the parabola again at the point $Q(aq^2, 2aq)$.

The line OP is perpendicular to the line OQ, where O is the origin.

Prove that $p^2 = 2$

(9)

$y^2 = 4ax$ implicit differentiation

$2y\frac{dy}{dx} = 4a$

$\therefore \frac{dy}{dx} = \frac{2a}{y} = \frac{2a}{2ap} = \frac{1}{p}$

normal at P : $y - 2ap = -p(x - ap^2)$

$y = -px + ap^3 + 2ap$

this normal passes through Q :

$2aq = -p(aq^2) + ap^3 + 2ap$

$\therefore \quad 2q = -pq^2 + p^3 + 2p \quad$ — ①

line OP : $y = \frac{2ap}{ap^2}x \rightarrow y = \frac{2}{p}x$

line OQ : $y = \frac{2aq}{aq^2}x \rightarrow y = \frac{2}{q}x$

now OP and OQ are perpendicular :

so $\frac{2}{p} \times \frac{2}{q} = -1 \qquad \therefore pq = -4$

so $q = \frac{-4}{p}$

① : $2\left(\frac{-4}{p}\right) = -p\left(\frac{16}{p^2}\right) + p^3 + 2p$

$\Rightarrow \frac{-8}{p} + \frac{16}{p} = p^3 + 2p$

$\times p$

$8 = p^4 + 2p^2$

$p^4 + 2p^2 - 8 = 0$

let $b = p^2$, $\quad b^2 + 2b - 8 = 0$

$(b+4)(b-2) = 0$

$b = -4 \quad$ or $\quad b = 2$

$p^2 = -4 \qquad \therefore p^2 = 2$

$p \in \mathbb{Z}$ so $p^2 \geq 0$

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Question 5 continued

(Total for Question 5 is 9 marks)

6. A tetrahedron has vertices $A(1, 2, 1)$, $B(0, 1, 0)$, $C(2, 1, 3)$ and $D(10, 5, 5)$.

Find

(a) a Cartesian equation of the plane ABC. **(3)**

(b) the volume of the tetrahedron $ABCD$. **(3)**

The plane Π has equation $2x - 3y + 3 = 0$

The point E lies on the line AC and the point F lies on the line AD.

Given that Π contains the point B, the point E and the point F,

(c) find the value of k such that $\overrightarrow{AE} = k\overrightarrow{AC}$. **(3)**

Given that $\overrightarrow{AF} = \frac{1}{9}\overrightarrow{AD}$

(d) show that the volume of the tetrahedron $ABCD$ is 45 times the volume of the tetrahedron $ABEF$. **(2)**

a) $\overrightarrow{AB} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$ $\quad \overrightarrow{AC} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$

$\therefore \; r \cdot \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = -3 + 2 + 2$

$r \cdot \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = 1$

$\hookrightarrow \boxed{-3x + y + 2z = 1}$

b) vol. $ABCD = \frac{1}{6}\left|\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \cdot (\overrightarrow{AD})\right|$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{pmatrix} 9 \\ 3 \\ 4 \end{pmatrix}$

$\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 3 \\ 4 \end{pmatrix} = -27 + 3 + 8 = -16$

$\therefore$ vol. $ABCD = \frac{1}{6}|-16| = \boxed{\frac{8}{3}}$

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Question 6 continued

c) Π : $2x - 3y + 3 = 0$

$\overrightarrow{AE} = k\overrightarrow{AC} = k\begin{pmatrix}1\\-1\\2\end{pmatrix}$

$\therefore \overrightarrow{OE} - \overrightarrow{OA} = \begin{pmatrix}k\\-k\\2k\end{pmatrix}$

$\overrightarrow{OE} = \overrightarrow{OA} + \begin{pmatrix}k\\-k\\2k\end{pmatrix} = \begin{pmatrix}k+1\\2-k\\2k+1\end{pmatrix}$

but OE lies on Π so $[2x - 3y + 3 = 0]$

$\Rightarrow 2(k+1) - 3(2-k) + 3 = 0$

$2k + 2 - 6 + 3k + 3 = 0$

$5k = 1$ $\therefore \boxed{k = \frac{1}{5}}$

d) volume ABEF $= \frac{1}{6}\left|(\overrightarrow{AB} \times \overrightarrow{AE}) \cdot (\overrightarrow{AF})\right|$

$\overrightarrow{AB} \times \overrightarrow{AE} = \begin{pmatrix}-1\\-1\\-1\end{pmatrix} \times \begin{pmatrix}1/5\\-1/5\\2/5\end{pmatrix} = \begin{pmatrix}-0.6\\0.2\\0.4\end{pmatrix}$

$\begin{pmatrix}-0.6\\0.2\\0.4\end{pmatrix} \cdot \begin{pmatrix}1\\1/3\\4/9\end{pmatrix} = -0.6 + \frac{1}{15} + \frac{8}{45}$

$= \frac{-16}{45}$

so vol ABEF $= \frac{1}{6}\left|\frac{-16}{45}\right|$

$= \frac{8}{3} \times \frac{1}{45}$

d) alt: vol ABEF $= \frac{1}{6}\left|(AB \times AE) \cdot AF\right|$

$= \frac{1}{6}\left|(AB \times \frac{1}{5}AC) \cdot (\frac{1}{9}AD)\right|$

$= \frac{1}{6}\left|\frac{1}{45}(AB \times AC).AD\right|$

$= \frac{1}{45}\left[\frac{1}{6}(AB \times AC).AD\right]$

(Total for Question 6 is 11 marks)

7. P and Q are two distinct points on the ellipse described by the equation $x^2 + 4y^2 = 4$

The line l passes through the point P and the point Q.

The tangent to the ellipse at P and the tangent to the ellipse at Q intersect at the point (r, s).

Show that an equation of the line l is

$$4sy + rx = 4$$

(8)

$x^2 + 4y^2 = 4$

$\Rightarrow 2x + 8y\left(\frac{dy}{dx}\right) = 0 \qquad \therefore \frac{dy}{dx} = \frac{-2x}{8y} = \frac{-x}{4y}$

$P\,(x_P, y_P)$

$Q\,(x_Q, y_Q)$

tangent at P : $y - y_P = \frac{-x_P}{4y_P}(x - x_P)$ — ①

at Q : $y - y_Q = \frac{-x_Q}{4y_Q}(x - x_Q)$ — ②

① : $\times 4y_P$: $4yy_P - 4y_P^2 = -xx_P + x_P^2$

$x_P^2 + 4y_P^2 = xx_P + 4yy_P$

but the equation of the ellipse is $[x^2 + 4y^2 = 4]$

so $x_P^2 + 4y_P^2 = 4 \quad (= xx_P + 4yy_P)$

similarly, $x_Q^2 + 4y_Q^2 = 4 \quad (= xx_Q + 4yy_Q)$

① and ② intersect at (r, s)

$①^* \Rightarrow xx_P + 4yy_P = 4$

$\Rightarrow rx_P + 4sy_P = 4$

$②^* \Rightarrow rx_Q + 4sy_Q = 4$

$\therefore rx_P + 4sy_P = rx_Q + 4sy_Q$

$4s(y_P - y_Q) = r(x_Q - x_P)$

$\therefore \frac{y_P - y_Q}{x_P - x_Q} = \frac{-r}{4s}$

equation of line l : $y - y_P = m(x - x_P)$

$m = \frac{y_P - y_Q}{x_P - x_Q} = \frac{-r}{4s}$

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Question 7 continued

so $y - y_p = \frac{-r}{4s}(x - x_p)$

$\times 4s$

$4sy - 4sy_p = -rx + rx_p$

so $4sy + rx = \underbrace{4sy_p + rx_p}_{①^*} = 4$

(Total for Question 7 is 8 marks)

8.

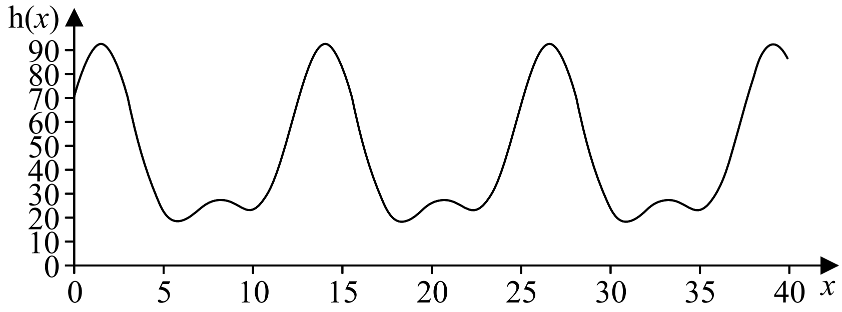


Figure 1

Figure 1 shows the graph of the function h(x) with equation

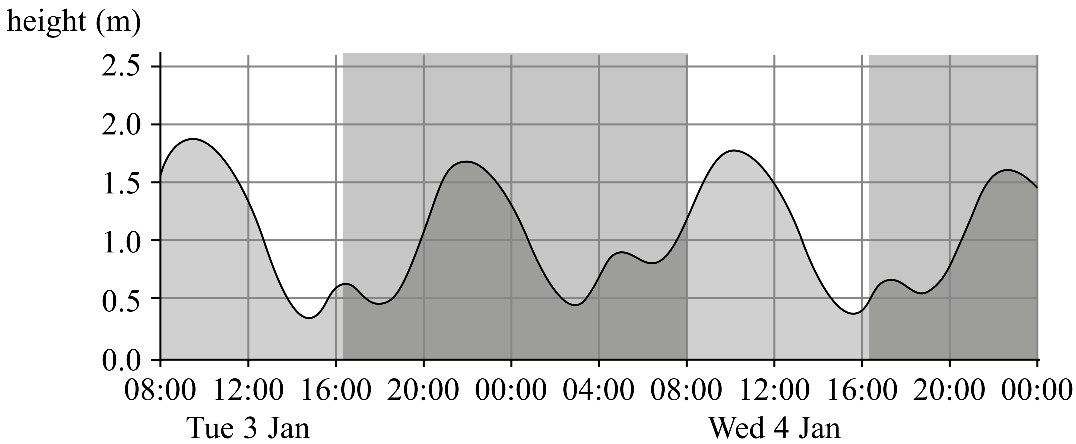
$$\mathrm{h}(x) = 45 + 15\sin x + 21\sin\left(\frac{x}{2}\right) + 25\cos\left(\frac{x}{2}\right) \qquad x \in [0, 40]$$

(a) Show that

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{(t^2 - 6t - 17)(9t^2 + 4t - 3)}{2(1 + t^2)^2}$$

where $t = \tan\left(\dfrac{x}{4}\right)$.

(6)



Source: [1]Data taken on 29th December 2016 from http://www.ukho.gov.uk/easytide/EasyTide

Figure 2

Figure 2 shows a graph of predicted tide heights, in metres, for Portland harbour from 08:00 on the 3rd January 2017 to the end of the 4th January 2017[1].

The graph of $k\mathrm{h}(x)$, where k is a constant and x is the number of hours after 08:00 on 3rd of January, can be used to model the predicted tide heights, in metres, for this period of time.

(b) (i) Suggest a value of k that could be used for the graph of $k\mathrm{h}(x)$ to form a suitable model.

(ii) Why may such a model be suitable to predict the times when the tide heights are at their peaks, but not to predict the heights of these peaks?

(3)

(c) Use Figure 2 and the result of part (a) to estimate, to the nearest minute, the time of the highest tide height on the 4th January 2017.

(6)

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Question 8 continued

a) $h(x) = 45 + 15\sin x + 21\sin\frac{x}{2} + 25\cos\frac{x}{2}$

$t = \tan\frac{x}{4}$: $\sin\frac{x}{2} = \frac{2t}{1+t^2}$

$\cos\frac{x}{2} = \frac{1-t^2}{1+t^2}$

$h'(x) = 15\cos x + \frac{21}{2}\cos\frac{x}{2} - \frac{25}{2}\sin\frac{x}{2}$

$\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1$

$= \frac{2(1-t^2)^2}{(1+t^2)^2} - 1$

$= \frac{2(1-t^2)^2 - (1+t^2)^2}{(1+t^2)^2}$

$\therefore h'(x) = 15\left[\frac{2(1-t^2)^2 - (1+t^2)^2}{(1+t^2)^2}\right] + \frac{21}{2}\left[\frac{1-t^2}{1+t^2}\right] - \frac{25}{2}\left[\frac{2t}{1+t^2}\right]$

$= \frac{60(1-2t^2+t^4) - 30(t^4+2t^2+1) + 21(1-t^2)(1+t^2) - 50t(1+t^2)}{2(1+t^2)^2}$

$\frac{60 - 30 - 120t^2 - 60t^2 + 60t^4 - 30t^4 + 21(1-t^4) - 50t - 50t^3}{2(1+t^2)^2}$

$\frac{(30t^4 - 21t^4) + 51 - 180t^2 - 50t^3 - 50t}{2(1+t^2)^2}$

$= \frac{9t^4 - 50t^3 - 180t^2 - 50t + 51}{2(1+t^2)^2}$

$= \frac{(t^2-6t-17)(9t^2+4t-3)}{2(1+t^2)^2}$

bi) At $h(0) = 70$, $m = 1.5$ on second graph at 8:00

$x = 0 \rightarrow h(x) = 70$

8:00 $\rightarrow$ height ≈ 1.5

$k \simeq \frac{1.5}{70}$

$\boxed{\simeq 0.0214}$

ii) The model from which this new graph is derived has approximately the same periodicity so it makes sense to use it to predict the times of the peaks.

However if you look at the first graph you can see that all the peaks are at a fixed

Question 8 continued

height. This isn't the case in the second graph so the model isn't suitable for predicting the heights of peaks.

c) no. of hours after 08:00 on 3 Jan $\simeq 26.5$

so $x \simeq 26.5$

remember that $h'(x) = \dfrac{(t^2-6t-17)(9t^2+4t-3)}{(1+t^2)^2}$

$h'(x) = 0 \quad : \quad 9t^2+4t-3=0$

$t = \dfrac{-2 \pm \sqrt{31}}{9}$

$t>0, \ t = \dfrac{-2+\sqrt{31}}{9}$

$x = 4\tan^{-1}\left(\dfrac{-2+\sqrt{31}}{9}\right)$

$x = 1.510, \ 1.51+\pi, \ 1.51+2\pi, \ldots$

we want the value of x (roughly) around 26.

so $x = 1.51 + 8\pi = 26.64$

closest value of x to 26

26.64 hours after 08:00 (Tue)

$\downarrow$

2.64 hours after 08:00 (wed)

$\downarrow$

$\boxed{10:39 \text{ AM}}$

(Total for Question 8 is 15 marks)

TOTAL FOR PAPER IS 75 MARKS

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